

On the multiple zeros of a partial theta function

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Abstract

We consider the partial theta function $\theta(q, x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j$, where $x \in \mathbb{C}$ is a variable and $q \in \mathbb{C}$, $0 < |q| < 1$, is a parameter. We show that, for any fixed q , if ζ is a multiple zero of the function $\theta(q, \cdot)$, then $|\zeta| \leq 8^{11}$.

The series $\theta(q, x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j$ in the variables q and x converges for $q \in \mathbb{D}_1 \setminus 0$, $x \in \mathbb{C}$, where \mathbb{D}_a stands for the open disk centered at the origin and of radius a . It defines a *partial theta function*. (We recall that the Jacobi theta function is the sum of the series $\Theta(q, x) := \sum_{j=-\infty}^{\infty} q^{j^2} x^j$ and the equality $\theta(q^2, x/q) = \sum_{j=0}^{\infty} q^{j^2} x^j$ holds true.) For any fixed q , θ is an entire function in x . We regard q as a parameter.

The function θ finds applications in statistical physics and combinatorics (see [14]), also in the theory of (mock) modular forms (see [3]), in asymptotic analysis (see [2]) and in Ramanujan-type q -series (see [15]). Its role in the framework of a problem concerning hyperbolic polynomials (i.e. real polynomials having all their zeros real) has been discussed in the papers [4], [13], [5], [12], [6], [11] and [7]). This problem has been studied earlier by Hardy, Petrovitch and Hutchinson (see [4], [5] and [13]). Other facts about the function θ can be found in [1] and [14].

Remark 1. It has been established in [10] that for any fixed value of the parameter q , the function θ has at most finitely-many multiple zeros. For $q \in (0, 1)$ there exists a sequence of values of q , tending to 1, for which $\theta(q, \cdot)$ has double real negative zeros tending to $-e^\pi$, see [8].

We prove the following theorem:

Theorem 2. *For any $q \in \mathbb{D}_1$, any multiple zero of θ belongs to the set $\overline{\mathbb{D}_{8^{11}}}$ ($8^{11} = 8589934592$).*

Lemma 3. *For any $q \in \overline{\mathbb{D}_{c_0}}$, $c_0 := 0.2078750206 \dots$, the function θ has no multiple zeros.*

(A similar result has been formulated independently by A. Sokal and J. Forsgård.)

Proof. Indeed, set $|x| = |q|^{-k-1/2}$, $k \in \mathbb{N}$. Then in the series of θ the term $L := x^k q^{k(k+1)/2}$ has the largest modulus (equal to $|q|^{-k^2/2}$). The sum M of the moduli of all other terms is smaller than $|q|^{-k^2/2} \tau(|q|)$, where $\tau := 2 \sum_{\nu=1}^{\infty} |q|^{\nu^2/2}$. The inequality $1 \geq \tau(|q|)$ is equivalent to $|q| \leq c_0$. Hence for $|q| \leq c_0$ one has $|L| > M$. Moreover, for no zero ζ of θ does one have $|\zeta| = |q|^{-k-1/2}$. For $|q| \leq 0.108$ all zeros ξ_k of θ are simple, see [9]. For any k fixed and for $|q|$ close to 0 one has $\xi_k \sim q^{-k}$ (see Proposition 10 in [7]). Hence for $|q| \leq c_0$ one has $|q|^{-k+1/2} < |\xi_k| < |q|^{-k-1/2}$, i.e. all zeros of θ are simple. \square

Proof of Theorem 2. We prove the theorem first in the case $1/2 \leq |q| < 1$. We use the fact that the Jacobi theta function Θ has only simple zeros (see [16]), so this is also true for the function $\Theta^*(q, x) = \Theta(\sqrt{q}, \sqrt{q}x) = \sum_{j=-\infty}^{\infty} q^{j(j+1)/2} x^j$. The zeros of $\Theta^*(q, x)$ are all simple and

equal $\mu_s := -1/q^s$, $s \in \mathbb{Z}$ (which can be deduced from the form of the zeros of Θ , see [16]). We recall that the Jacobi triple product is the equality $\Theta(q, x^2) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + x^2 q^{2m-1})(1 + x^{-2} q^{2m-1})$ (see [16]) from which follows the identity $\Theta^*(q, x) = \prod_{m=1}^{\infty} (1 - q^m)(1 + x q^m)(1 + x^{m-1}/x)$.

Notation 4. Set $G := \sum_{j=-\infty}^{-1} q^{j(j+1)/2} x^j$. Thus $\theta = \Theta^* - G$. For given x ($|x| > 1$) and q we denote by κ the least value of $m \in \mathbb{N}$ for which one has $|x q^m| < 1$. Set $Q := \prod_{m=1}^{\infty} (1 - q^m)$, $R := \prod_{m=1}^{\infty} (1 + q^{m-1}/x)$ and $U_p^s := \prod_{m=p}^s (1 + x q^m)$, $s \geq p$. By $\mathcal{C}(v, r)$, $v \in \mathbb{C}$, $r > 0$, we denote the circumference (in the x -space) about v and of radius r and by $\mathcal{D}(v, r)$ the corresponding open disk. We set $X_\rho := \{x \in \mathbb{C}, |x| > \rho, \rho > 0\}$.

Remarks 5. (1) Suppose that $1 - 1/(n-1) \leq |q| \leq 1 - 1/n$, $n = 3, 4, \dots$. For $s_1 > s_2 > 0$ one has $|\mu_{s_1}| > 1$, $|\mu_{s_2}| > 1$ and $|\mu_{s_1} - \mu_{s_2}| > 1/n$ (because $|1/q| \geq 1/(1 - 1/n) > 1 + 1/n$). Hence the two closed disks $\overline{\mathcal{D}(\mu_{s_i}, 1/2n)}$, $i = 1, 2$, do not intersect.

(2) For $x \in X_\rho$, $\rho > 1$, one has $|G| \leq \sum_{j=-\infty}^{-1} \rho^j = 1/(\rho - 1)$.

Proposition 6. Suppose that $1 - 1/(n-1) \leq |q| \leq 1 - 1/n$, $n = 3, 4, \dots$, and that for a given $s \in \mathbb{N}$ the circumference $\mathcal{C}(\mu_s, 1/2n)$ (hence the closed disk $\mathcal{D}(\mu_s, 1/2n)$ as well) belongs to the set $X_{8^{11}}$. Then at any point of this circumference one has $|\Theta^*| > 1 > 1/(8^{11} - 1) \geq |G|$.

Before proving Proposition 6 we deduce Theorem 2 from it. By the Rouché theorem the functions Θ^* and θ have one and the same number of zeros (counted with multiplicity) inside $\mathcal{C}(\mu_s, 1/2n)$. For Θ^* this number is 1, hence θ has a single zero, a simple one, inside $\mathcal{C}(\mu_s, 1/2n)$. For any fixed $s \in \mathbb{N}$ and for $|q|$ sufficiently small ($q \neq 0$) the function $\theta(q, \cdot)$ has a zero ξ_s close to μ_s (close in the sense that $(\xi_s - \mu_s) \rightarrow 0$ as $q \rightarrow 0$, see [7]). Hence this is the simple zero inside $\mathcal{C}(\mu_s, 1/2n)$. For $0 < |q| \leq 0.108$ the numbers ξ_s are all the zeros of θ (see [9]); these zeros are simple. As $|q|$ increases, for certain values of q a confluence of certain zeros occurs (see [11]).

Fix $s \in \mathbb{N}$. If for $0 < |q| = \alpha \leq 1 - 1/n$ one has $\mathcal{C}(\mu_s, 1/2n) \subset X_{8^{11}}$, then this inclusion holds true for $0 < |q| \leq \alpha$ as well. This means that for $0 < |q| \leq 1 - 1/n$ the zeros ξ_k of θ with $k \geq s$ remain distinct, simple and belong to the interiors of the respective circumferences $\mathcal{C}(\mu_k, 1/2n)$. Hence for $1 - 1/(n-1) \leq |q| \leq 1 - 1/n$ and $|x| > 8^{11}$ there are no multiple zeros of θ . This is true for any $n = 3, 4, \dots$. Hence for $|x| > 8^{11}$ and $|q| \in [1/2, 1)$, the function θ has no multiple zeros.

In the proof of Proposition 6 we use the following lemma:

Lemma 7. Suppose that $|q| \leq 1 - 1/b$, $b > 1$, and $|x| > 1$. Then $|Q| \geq e^{(\pi^2/6)(1-b)}$, $|R| \geq (1 - 1/|x|)e^{(\pi^2/6)(1-b)}$ and $|U_{\kappa+1}^\infty| \geq e^{(\pi^2/6)(1-b)}$.

Proof. Indeed, $|Q| \geq S := \prod_{m=1}^{\infty} (1 - |q|^m)$. Hence

$$\begin{aligned} \ln S &= -\sum_{m=1}^{\infty} |q|^m - (1/2) \sum_{m=1}^{\infty} |q|^{2m} - (1/3) \sum_{m=1}^{\infty} |q|^{3m} - \dots \\ &= -|q|/(1 - |q|) - |q|^2/2(1 - |q|^2) - |q|^3/3(1 - |q|^3) - \dots \\ &= (-|q|/(1 - |q|))T, \quad \text{where } T = 1 + |q|/2(1 + |q|) + |q|^2/3(1 + |q| + |q|^2) + \dots \end{aligned}$$

Clearly $-|q|/(1 - |q|) = 1 - 1/(1 - |q|) \in (1 - b, 0)$ and $|q|^s/(s+1)(1 + |q| + \dots + |q|^s) < 1/(s+1)^2$. Hence $T \in (0, \sum_{s=0}^{\infty} 1/(s+1)^2 = \pi^2/6 = 1.6449\dots)$ and $|Q| \geq S \geq e^{(\pi^2/6)(1-b)}$.

To obtain the second (resp. the third) inequality just observe that for $m \geq 2$ one has $|1 + q^{m-1}/x| \geq 1 - |q|^{m-1}/|x| > 1 - |q|^{m-1}$ (resp. that for $m \geq \kappa + 1$ one has $|1 + x q^m| \geq 1 - |x q^\kappa| |q|^{m-\kappa} > 1 - |q|^{m-\kappa}$) and then apply the first inequality. \square

Proof. Proof of Proposition 6.

(A) One has $1/4 \leq (1 - 1/(n-1))^{n-1} \leq 1/e$ and $1/8 \leq (1 - 1/(n-1))^n \leq 1/e$.

(B) Suppose that $|x| > 8^{11}$. Then $\kappa > 11n$. Indeed, as $1 - 1/n \geq |q| \geq 1 - 1/(n-1)$, one has $1/e^{11} \geq (1 - 1/n)^{11n} \geq |q|^{11n} \geq (1 - 1/(n-1))^{11n} \geq 1/8^{11}$. Hence $|x||q|^{11n} \geq |x|/8^{11} > 1$ which implies $\kappa > 11n$.

(C) Consider the product U_1^κ . For a point external for all open disks $\mathcal{D}(\mu_i, 1/2n)$, $i \in \mathbb{N}$, one has $|xq^\kappa + 1| = |q^\kappa||x - \mu_\kappa| \geq |q^\kappa|/2n$ and $|xq^{\kappa-1} + 1| = |q^{\kappa-1}||x - \mu_{\kappa-1}| \geq |q^{\kappa-1}|/2n$. As $\kappa > 8$ and $|x| \geq |q|^{1-\kappa}$, for $\nu = 1, \dots, 4$ one has $|xq^\nu + 1| \geq |x||q|^\nu(1 - |xq^\nu|^{-1}) > |x||q|^\nu(1 - |x|^{-1/2})$, so

$$\begin{aligned} |xq^\kappa + 1||xq + 1||xq^2 + 1| &\geq |q^\kappa||x|^2|q|^3(1 - |x|^{-1/2})^2/2n \geq (1 - |x|^{-1/2})^2/2n \\ \text{and} \\ |xq^{\kappa-1} + 1||xq^3 + 1||xq^4 + 1| &\geq |q^{\kappa-1}||x|^2|q|^7(1 - |x|^{-1/2})^2/2n \geq (1 - |x|^{-1/2})^2/2n. \end{aligned}$$

(D) For $m \leq \kappa - 2$ one has $|x||q|^m > 1$ and $|1 + xq^m| \geq |x||q|^m(1 - |q|^{\kappa-1-m}/|xq^{\kappa-1}|) \geq (1 - |q|^{\kappa-1-m})$. Suppose that $l \in \mathbb{N} \cup 0$, $\kappa - 3 \geq l \geq 0$. By analogy with Lemma 7 one can show that $|U_{\kappa-2-l}^{\kappa-2}| \geq e^{(\pi^2/6)(1-n)}$.

(E) Consider the product $|U_{\kappa-2-4n}^{\kappa-2}| = \prod_{m=\kappa-2-4n}^{\kappa-2} |x||q|^m|1 + q^{\kappa-1-m}/xq^{\kappa-1}|$. The largest of the factors $|x||q|^m|$ is obtained for $m = \kappa - 2 - 4n$. It equals $|x||q|^{\kappa-2}|q|^{-4n} > |q|^{-4n} > e^4 > 8^{\pi^2/6} + 1$ (because $8^{\pi^2/6} = 30.5 \dots < e^{3.5} = 33.1 \dots$). Thus for $m < \kappa - 2 - 4n$ the inequalities $|1 + xq^m| \geq |xq^m| - 1 > 8^{\pi^2/6}$ hold true.

(F) To prove the proposition it remains to show that $|\Theta^*| = |Q||R||U_1^\infty| > 1$. Set $U_1^\infty = U_{\kappa+1}^\infty(1+xq^\kappa)(1+xq^{\kappa-1})U_{\kappa-2-4n}^{\kappa-2}U_5^{\kappa-3-4n}U_1^4$. It was shown in (C) that $|(1+xq^\kappa)(1+xq^{\kappa-1})U_1^4| \geq (1 - |x|^{-1/2})^4/4n^2$ (*). As $\kappa > 11n$, there are at least $5n$ factors in the product $U_5^{\kappa-3-4n}$, and by (E) their moduli are $> 8^{\pi^2/6}$. Denote by P_1 and P_2 the products respectively of $4n$ and n of these factors (assumed all distinct). Using Lemma 7 one finds that $|P_1||Q||R||U_{\kappa+1}^\infty||U_{\kappa-2-4n}^{\kappa-2}| \geq (8^{\pi^2/6})^{4n}(1 - |x|^{-1})e^{4(\pi^2/6)(1-n)} > 1$ (**) (because $e < 8$ and $(1 - |x|^{-1})e^{4(\pi^2/6)} > 1$) and $|P_2|(1 - |x|^{-1/2})^4/4n^2 > 1$ (***) (***). Thus Proposition 6 follows from inequalities (*), (**) and (***). \square

Now we prove Theorem 2 for $c_0 \leq |q| \leq 1/2$. Lemma 7 implies that for $c_0 \leq |q| \leq 1/2$ and $|x| > 8^{11}$ one has $|Q| \geq c_1 := S|_{|q|=1/2} = 0.2887880950$, $|R| \geq (1 - |x|^{-1})c_1 > 0.2887880949 =: c_2$ and $|U_{\kappa+1}^\infty| \geq c_1$. Indeed, $|Q| \geq S$ and S is minimal for $|q| = 1/2$.

We need to modify the proof of Proposition 6 so that it should become valid also for $c_0 \leq |q| < 1/2$. We observe first that $\kappa \geq 15$, with equality for $|x| = 8^{11}$, $|q| = c_0$. Instead of the disks $\mathcal{D}(\mu_i, 1/2n)$ we consider the disks $\mathcal{D}(\mu_i, 1/4)$; their respective radii are defined by the conditions $|q| \leq 1 - 1/n$ and $|q| \leq 1 - 1/2$, see part (1) of Remarks 5. Thus the displayed inequalities of part (C) of the proof of the proposition and inequality (*) of part (F) remain valid with n replaced by 2.

Set $U_1^\infty = U_{\kappa+1}^\infty(1+xq^\kappa)(1+xq^{\kappa-1})U_5^{\kappa-2}U_1^4$. The factor $U_5^{\kappa-2}$ contains at least 9 factors and their respective moduli are not less than $8^{11}c_0^s - 1$, $s = 5, \dots, 13$. Thus

$$\begin{aligned} |\Theta^*| &\geq |Q||R||U_{\kappa+1}^\infty||1+xq^\kappa(1+xq^{\kappa-1})U_1^4||U_5^{\kappa-2}| \\ &\geq c_1c_2c_1(1/16)(1 - 8^{-5.5})^4 \prod_{s=5}^{13} (8^{11}c_0^s - 1) > 1. \quad \square \end{aligned}$$

\square

Remark 8. The number 8^{11} in the formulation of the theorem seems not to be optimal. The optimal number is not less than e^π , see Remark 1.

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